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Comments on Quiver Gauge Theories and Matrix Model

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abstract

Dijkgraaf and Vafa have conjectured the correspondences between topological string theories, $\mathcal{N} = 1$ gauge theories and matrix models. By the use of this conjecture, we calculate the quantum deformations of Calabi-Yau threefolds with ADE singularities from ADE multi-matrix models. We obtain the effective superpotentials of the dual quiver gauge theories in terms of the geometric engineering for the deformed geometries. We find the Veneziano-Yankielowicz terms in the effective superpotentials.

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1. Introduction

String theories give us a lot of useful methods in order for us to understand various gauge theories. For example, the AdS/CFT correspondence [1] and the gauge/gravity correspondence [2] are known well. A-model topological string theories correspond to Chern-Simons gauge theories by the gauge/gravity correspondence.

There are mirror symmetry between the A-model topological string theories and the B-model topological string theories. Recently Dijkgraaf and Vafa have proposed the correspondences between the B-model topological string theories, $\mathcal{N}=1$ supersymmetric gauge theories and large N matrix models [3–5]. In other words, these correspondences are the mirror dual of the gauge/gravity correspondence. The $\mathcal{N}=1$ gauge theory is constructed by adding certain superpotential to the $\mathcal{N}=2$ gauge theory. In the description of D-brane configuration, the $\mathcal{N}=1$ gauge theory is realized by D5-branes wrapped on two-cycles in Calabi-Yau manifolds. When the two-cycles are blown down, new three-cycles emerge. Three-form RR and NSNS fluxes then appear instead of the D5-branes. This is called geometric transition [6–9]. On the Calabi-Yau manifolds after the geometric transition, there are two kinds of three cycles, which are compact A_i-cycles and non-compact B_i-cycles. We define periods,

$$\mu_i = \frac{1}{2\pi i}\oint_{A_i}\Omega, \quad \Pi_i = \frac{\partial \mathcal{F}_0}{\partial \mu_i} = \int_{B_i}\Omega, \tag{1.1}$$

where Ω is a holomorphic three-form on the Calabi-Yau three-fold and $\mathcal{F}_0$ is a prepotential. In terms of these periods we can write down the effective superpotential of the dual gauge theory,

$$W_{\text{eff}} = \sum_i (N_i\Pi_i - 2\pi i\tau_i\mu_i), \tag{1.2}$$

where N_i is a number of D-branes and τ_i is a gauge coupling.

It is proposed that the effective superpotential can be reproduced by the matrix model with certain tree-level superpotential $W(\Phi)$ [3]. The partition function of the matrix model is $Z = \int d\Phi \exp\left(-\frac{1}{g_s}W(\Phi)\right)$. Fixing $S_i = N_i g_s$, we take the limit $N_i \gg 1, g_s \ll 1$, and the partition function then leads to $Z = \exp\sum_g g_s^{2g-2}\mathcal{F}_g(S_i)$. The free energies $\mathcal{F}_g$ are the contributions of genus g diagrams. In particular $\mathcal{F}_0$ is derived from the planar diagrams. If we can calculate the partition function, we obtain the free energy, $\log Z$. We can identify it with the prepotential of the dual gauge theory and we obtain the exact effective superpotential in terms of (1.1) and (1.2). Since the perturbative analyses in the matrix models lead to the exact results in the dual gauge theories, this new derivation of the superpotentials is powerful. A lot of works on this subject have been done [10–44].

In this paper we consider $\mathcal{N} = 1$ quiver gauge theories and matrix models. The matrix models are ADE multi-matrix models, which have been studied in [45,46]. The quiver gauge theories are realized by the string theories on the Calabi-Yau manifolds with ADE singularities. The $\mathcal{N} = 2$ quiver gauge theories lead to the $\mathcal{N} = 1$ theories by the additional superpotentials, while the dual Calabi-Yau geometries are deformed. These deformations are reproduced by the matrix model [4]. Since we systematically introduce a lot of gauge symmetries to the quiver gauge theories, they are interesting also for realistic particle theories [22].

In Section 2, we will analyse the quantum deformations of the ADE singularities in the matrix model side. In terms of the deformed geometries, we will calculate the superpotentials of the dual quiver gauge theories. Section 3 is devoted to the conclusions and the some comments on left problems.

2. Effective superpotentials of quiver gauge theories

Before discussing on the quiver gauge theories and the multi-matrix models, we will give a brief review on a one-matrix model [3]. Let us consider an $N \times N$ Hermitean matrix Φ. The partition function of the one-matrix model with the tree-level superpotential $W(\Phi)$ is

$$Z = \int d\Phi \exp\left(-\frac{1}{g_s} W(\Phi)\right), \tag{2.1}$$

where we set that $W(\Phi)$ is a degree n polynomial of Φ. In terms of the N eigen values λ_I $(I = 1, \cdots, N)$ of Φ, we can rewrite the partition function (2.1) as

$$Z = \int \left(\prod_{I=1}^{N} d\lambda_I\right) \Delta(\lambda)^2 \exp\left(-\frac{1}{g_s} \sum_{I=1}^{N} W(\lambda_I)\right), \tag{2.2}$$

where $\Delta(\lambda)$ is the Vandermonde determinant, $\prod_{I<J}(\lambda_I - \lambda_J)$. When we describe the partition function as $Z = \int \Pi_I d\lambda_I e^{-\hat{S}}$, the effective action $\hat{S}$ is denoted by

$$\hat{S} = \frac{1}{g_s} \sum_{I=1}^{N} W(\lambda_I) - 2 \sum_{I<J} \log(\lambda_I - \lambda_J). \tag{2.3}$$

From the action (2.3), the equations of motion for λ_I are written down as

$$\frac{1}{g_s}W'(\lambda_I) - 2\sum_{J\neq I}\frac{1}{\lambda_I-\lambda_J} = 0. \tag{2.4}$$

We introduce a resolvent,

$$\omega(x) \equiv \frac{1}{N}\sum_{I=1}^{N}\frac{1}{x-\lambda_I}, \tag{2.5}$$

which is useful in the matrix model technology [47,48]. Physical meaning of the resolvent is a loop operator and we can easily derive a loop equation from (2.4) in terms of the resolvent. From (2.4), we define the function,

$$y(x) = W'(x) - 2g_s\sum_{J=1}^{N}\frac{1}{x-\lambda_J} = W'(x) - 2S\omega(x), \tag{2.6}$$

where S is the 't Hooft coupling Ng_s. $\omega(x)$ is not a polynomial, but, in large N limit, $y(x)^2$ is given by $y(x)^2 = W'(x)^2 + f_{n-1}(x)$, where $f_{n-1}(x)$ is a degree $n-1$ polynomial.

In the context of large N duality and geometric transitions [6–9], the dual Calabi-Yau geometry after the deformation is denoted by

$$u^2 + v^2 + y^2 + W'(x)^2 + f_{n-1}(x) = 0. \tag{2.7}$$

We then consider the one-form

$$y(x)dx = \sqrt{W'(x)^2 + f_{n-1}(x)}dx. \tag{2.8}$$

The periods (1.1) are described as

$$\frac{1}{2\pi i}\int_{A_i} y(x)dx = \mu_i, \quad \int_{B_i}^{\Lambda^{\frac{3}{2}}} y(x)dx = \frac{\partial\mathcal{F}_0}{\partial\mu_i} = \Pi_i, \tag{2.9}$$

in terms of the one-form (2.8). Without the deformation, $y(x)$ in (2.8) is equal to $W'(x)$. Since the function (2.6) derived from the matrix model can be identified with $y(x)$ in (2.8), $-2S\omega(x)$ in (2.6) leads to $f_{n-1}(x)$ in (2.7), in other words, $f_{n-1}(x)$ is regarded as the contribution of loop operators in the matrix model. Adding f_{n-1} is called a quantum deformation.

Let us now consider the ADE singularities and the quiver gauge theories. The Calabi-Yau three-folds with the ADE singularities are realized by the fibration of two dimensional ADE singularities. The fibres over x-plane are denoted by

$$A_n \quad u^2 + v^2 + \prod_{i=1}^{n+1}(y - t_i(x)) = 0, \quad \sum_{i=1}^{n+1} t_i = 0, \tag{2.10}$$

$$D_n \quad u^2 + v^2 y + \frac{1}{y}\left(\prod_{i=1}^{n}(y - t_i(x)^2) - \prod_{i=1}^{n} t_i(x)^2\right) + 2\prod_{i=1}^{n} v t_i(x) = 0, \tag{2.11}$$

$$E_6 \quad u^2 + v^3 + y^4 + \epsilon_2(x)vy^2 + \epsilon_5(x)vy + \epsilon_6(x)y^2 + \epsilon_8(x)v$$
$$+ \epsilon_9(x)y + \epsilon_{12}(x) = 0, \tag{2.12}$$

$$E_7 \quad u^2 + v^3 + vy^3 + \epsilon_2(x)v^2y + \epsilon_6(x)v^2 + \epsilon_8(x)vy + \epsilon_{10}(x)y^2$$
$$+ \epsilon_{12}(x)v + \epsilon_{14}(x)y + \epsilon_{18}(x) = 0, \tag{2.13}$$

$$E_8 \quad u^2 + v^3 + y^5 + \epsilon_2(x)vy^3 + \epsilon_8(x)vy^2 + \epsilon_{12}(x)y^3 + \epsilon_{14}(x)vy$$
$$+ \epsilon_{18}(x)y^2 + \epsilon_{20}(x)v + \epsilon_{24}(x)y + \epsilon_{30}(x) = 0, \tag{2.14}$$

where ϵ_i are functions of $t_i(x)$ and are explicitly written down in [49]. For these fibrations we can describe the one-forms $y_i(x)dx$ [8] as

$$A_n \quad y_i = t_{i+1} - t_i, \quad i = 1, \cdots, n, \tag{2.15}$$

$$D_n \quad y_i = t_{i+1} - t_i, \quad i = 1, \cdots, n-1, \qquad y_n = -t_{n-1} - t_n, \tag{2.16}$$

$$E_n \quad y_i = t_{i+1} - t_i, \quad i = 1, \cdots, n-1, \qquad y_n = t_1 + t_2 + t_3. \tag{2.17}$$

If we calculate the periods (2.9) for the one-forms (2.15), (2.16) and (2.17), we obtain the effective superpotentials of the quiver gauge theories.

In the following, we will derive the quantum deformations of the ADE singularities from the ADE matrix models and consider the effective superpotentials of the dual quiver gauge theories.

2.1. A_n singularity

Firstly we consider the A_n singularities and the A_n quiver gauge theories.

For the simplest example, let us study an A_2 singularity [4]. The A_2 quiver diagram is denoted by the A_2 Dynkin diagram represented in fig. 1. We assign a $U(N_i)$ gauge group to the i-th node of the A_2 diagram. The dual quiver gauge theory consists of the adjoint scalars Φ_1, Φ_2 and the bifundamental matters Q_{12}, Q_{21} transforming in the representation of $(N_1, \bar{N}_2), (N_2, \bar{N}_1)$ respectively. The tree-level superpotential is

$$W(\Phi, Q) = \mathrm{Tr} Q_{12}\Phi_2 Q_{21} - \mathrm{Tr} Q_{21}\Phi_1 Q_{12} + W_1(\Phi_1) + W_2(\Phi_2).$$

The supersymmetry of the quiver gauge theory is broken from $\mathcal{N}=2$ to $\mathcal{N}=1$ by inserting the superpotentials $W_i(\Phi_i)$. The partition function of the matrix model which we should consider is $Z = \int d\Phi dQ \exp\left(-\frac{1}{g_s} W(\Phi, Q)\right)$. Φ_i is an $N_i \times N_i$ matrix and Q_{ij} is an $N_i \times N_j$ matrix. Integrating Q_{ij} out in the partition function, we obtain the effective action of Φ_i. Since Φ_i is the $N_i \times N_i$ matrix, using the eigen values $\lambda_{i,I}$ $(i = 1, 2, \quad I = 1, \cdots, N_i)$, we exchange the matrix integrals $\int d\Phi$ for the eigen value integrals $\int d\lambda_{i,I}$. We then obtain the equations of motion [4],

$$\frac{1}{g_s} W_1'(\lambda_{1,I}) - 2 \sum_{J=1, J\neq I}^{N_1} \frac{1}{\lambda_{1,I} - \lambda_{1,J}} + \sum_{J=1}^{N_2} \frac{1}{\lambda_{1,I} - \lambda_{2,J}} = 0, \tag{2.18}$$

$$\frac{1}{g_s} W_2'(\lambda_{2,I}) - 2 \sum_{J=1, J\neq I}^{N_2} \frac{1}{\lambda_{2,I} - \lambda_{2,J}} + \sum_{J=1}^{N_1} \frac{1}{\lambda_{2,I} - \lambda_{1,J}} = 0. \tag{2.19}$$

The second terms in (2.18) and (2.19) are the contributions of loop effects of $\lambda_{i,I}$, and the third terms come from the contributions of Q_{ij}. On the analogy of (2.5), we define resolvents,

$$\omega_1(x) \equiv \frac{1}{N_1} \sum_{I=1}^{N_1} \frac{1}{x - \lambda_{1,I}}, \quad \omega_2(x) \equiv \frac{1}{N_2} \sum_{I=1}^{N_2} \frac{1}{x - \lambda_{2,I}}. \tag{2.20}$$

By the way, the classical deformation of the A_2 singularity is denoted by

$$u^2 + v^2 + (y - t_1^{\mathrm{cl}}(x))(y - t_2^{\mathrm{cl}}(x))(y - t_3^{\mathrm{cl}}(x)) = 0, \quad \sum_{i=1}^{3} t_i^{\mathrm{cl}}(x) = 0, \tag{2.21}$$

where the deformation parameters $t_i^{\mathrm{cl}}(x)$ are given by

$$t_1^{\mathrm{cl}}(x) = -\frac{2W_1'(x) + W_2'(x)}{3}, \quad t_2^{\mathrm{cl}}(x) = \frac{W_1'(x) - W_2'(x)}{3}, \quad t_3^{\mathrm{cl}}(x) = \frac{W_1'(x) + 2W_2'(x)}{3}, \tag{2.22}$$

from (2.10) and (2.15)[8]. On the other hand, from (2.18), (2.19) and (2.20), we obtain the one-form $y_i(x)dx$ which are described as

$$y_1(x) = W_1'(x) - 2S_1\omega_1(x) + S_2\omega_2(x), \quad y_2(x) = W_2'(x) + S_1\omega_1(x) - 2S_2\omega_2(x). \tag{2.23}$$

Since the classical deformations are given by $y_1^{\rm cl}(x) = t_2^{\rm cl} - t_1^{\rm cl} = W_1'(x)$ and $y_2^{\rm cl}(x) = t_3^{\rm cl} - t_2^{\rm cl} = W_2'(x)$, the terms including ω_i in (2.23) imply quantum effects. We can define the quantum deformation of the A_2 singularity [4] as

$$u^2 + v^2 + (y - t_1(x))(y - t_2(x))(y - t_3(x)) = 0, \quad \sum_{i=1}^{3} t_i(x) = 0, \tag{2.24}$$

so that $t_i(x)$ satisfy $y_1(x) = t_2(x) - t_1(x)$ and $y_2(x) = t_3(x) - t_2(x)$. Actually $t_i(x)$ can be written down as

$$t_1(x) = t_1^{\rm cl}(x) + S_1\omega_1(x), \quad t_2(x) = t_2^{\rm cl}(x) - S_1\omega_1(x) + S_2\omega_2(x), \quad t_3(x) = t_3^{\rm cl}(x) - S_2\omega_2(x). \tag{2.25}$$

So far we have given a brief review on the geometry of the A_2 quiver [4]. We will generalize the above discussions to the A_n quiver and calculate the effective superpotentials of A_n quiver gauge theories. We will consider, in particular, the quadratic tree-level superpotentials.

The A_n quiver diagram is represented in fig. 2. Since the diagram has n nodes, we assign a $U(N_i)$ gauge group to each i-th node. The dual theory is the A_n quiver gauge theory which consists of the adjoint scalars Φ_i and the bifundamental matters $Q_{i,i+1}, Q_{i+1,i}$. The tree-level superpotential is

$$W(\Phi, Q) = \sum_{i=1}^{n-1} \mathrm{Tr}(Q_{i,i+1}\Phi_{i+1}Q_{i+1,i} - Q_{i+1,i}\Phi_i Q_{i,i+1}) + \sum_{i=1}^{n} \mathrm{Tr} W_i(\Phi_i),$$

where $W_i(\Phi_i)$ is a polynomial of Φ_i. We integrate out Q_{ij} in the partition function,

$$Z = \int \prod_{i=1}^{n} d\Phi_i \prod_{i,j} dQ_{ij} \exp\left(-\frac{1}{g_s} W(\Phi, Q)\right),$$

and we can rewrite it in terms of the eigen values $\lambda_{i,I}$ $(I = 1, \cdots, N_i)$ of Φ_i. We then obtain

$$\begin{aligned} Z &= \int \prod_{i=1}^{n} d\Phi_i \frac{1}{\prod_i \det(\Phi_i \otimes \mathbf{1} - \mathbf{1} \otimes \Phi_{i+1})} \exp\left(-\frac{1}{g_s} W(\Phi, Q)\right) \\ &= \int \prod_{i,I} d\lambda_{i,I} \frac{\prod_{i,I<J}(\lambda_{i,I} - \lambda_{i,J})^2}{\prod_{i,I,J}(\lambda_{i,I} - \lambda_{i+1,J})} \exp\left(-\frac{1}{g_s} \sum_{i,I} W_i(\lambda_{i,I})\right). \end{aligned} \tag{2.26}$$

When we describe the partition function as $\int \prod_{i,I} d\lambda_{i,I} e^{-\hat{S}}$, the effective action $\hat{S}$ is denoted by

$$\hat{S} = \frac{1}{g_s} \sum_{i,I} W_i(\lambda_{i,I}) - 2 \sum_{i,I<J} \log(\lambda_{i,I} - \lambda_{i,J}) + \sum_{i,I,J} \log(\lambda_{i,I} - \lambda_{i+1,J}). \tag{2.27}$$

The equations of motion for the action (2.27) become

$$\begin{aligned} &\frac{1}{g_s} W_1'(\lambda_{1,I}) - 2 \sum_{J=1,J\neq I}^{N_1} \frac{1}{\lambda_{1,I} - \lambda_{1,J}} + \sum_{J=1}^{N_2} \frac{1}{\lambda_{1,I} - \lambda_{2,J}} = 0, \\ &\frac{1}{g_s} W_j'(\lambda_{j,I}) - 2 \sum_{J=1,J\neq I}^{N_j} \frac{1}{\lambda_{j,I} - \lambda_{j,J}} \\ &+ \sum_{J=1}^{N_{j-1}} \frac{1}{\lambda_{j,I} - \lambda_{j-1,J}} + \sum_{J=1}^{N_{j+1}} \frac{1}{\lambda_{j,I} - \lambda_{j+1,J}} = 0, \quad j = 2, \cdots, n-1, \\ &\frac{1}{g_s} W_n'(\lambda_{n,I}) - 2 \sum_{J=1,J\neq I}^{N_n} \frac{1}{\lambda_{n,I} - \lambda_{n,J}} + \sum_{J=1}^{N_{n-1}} \frac{1}{\lambda_{n,I} - \lambda_{n-1,J}} = 0. \end{aligned} \tag{2.28}$$

We also introduce the resolvents,

$$\omega_i(x) \equiv \frac{1}{N_i}\sum_{I=1}^{N_i}\frac{1}{x-\lambda_{i,I}}, \quad i=1,\cdots,n, \tag{2.29}$$

and the 't Hooft couplings $S_i \equiv g_s N_i$. Note that, in the context of the large N duality, we fix S_i and take the limits where N_i go to infinity. We then read the following functions from (2.28) as

$$y_1(x) = W_1'(x) - 2S_1\omega_1(x) + S_2\omega_2(x), \tag{2.30}$$

$$y_j(x) = W_j'(x) - 2S_j\omega_j(x) + S_{j-1}\omega_{j-1}(x) + S_{j+1}\omega_{j+1}(x), \quad j=2,\cdots,n-1, \tag{2.31}$$

$$y_n(x) = W_n'(x) - 2S_n\omega_n(x) + S_{n-1}\omega_{n-1}(x). \tag{2.32}$$

$y_i(x)$ include the quantum deformations of A_n singularity and $y_i(x)dx$ denote the deformed one-forms. Note that $y_i^{\rm cl} = W_i'$ are regarded as the classical deformations.

Let us set the tree-level superpotentials to be the quadratic ones,

$$W_i(x) = \frac{m_i}{2}x^2, \quad i=1,\cdots,n. \tag{2.33}$$

From $W_i'(\lambda_{i,I}) = 0$, the classical vacua are $\lambda_{i,I}=0$. Since the perturbative analyses around these vacua in the matrix model give us the exact effective superpotentials of the $\mathcal{N}=1$ dual gauge theories, we approximate $\lambda_{i,I}$ to the vacuum expectation values, that is, we set all $\lambda_{i,I}$ to be equal to zero. (2.30) then becomes

$$y_1(x) = m_1 x - \frac{2S_1 - S_2}{x} = \frac{m_1}{x}(x-a_1^+)(x-a_1^-),$$

where $a_1^\pm = \pm\sqrt{\frac{2S_1-S_2}{m_1}}$. The critical point $a_1=0$ of $W_1'(a_1)=0$ is splitted to the two points $a_1^\pm$. In the same way, each critical point $a_i=0$ $(i=2,\cdots,n-1)$ is splitted to the points $a_i^\pm = \pm\sqrt{\frac{2S_i-S_{i-1}+S_{i+1}}{m_i}}$, and $a_n=0$ is splitted to $a_n^\pm = \pm\sqrt{\frac{2S_n-S_{n-1}}{m_n}}$. In other words, every original critical point is resolved to the two points. In terms of these resolved points, the periods (2.9) are described as

$$\mu_i = \frac{1}{2\pi i}\int_{a_i^-}^{a_i^+} y_i(x)dx, \quad \Pi_i = \int_{a_i^+}^{\Lambda_i^{\frac{3}{2}}} y_i(x)dx. \tag{2.34}$$

Since the B_i-cycles are non-compact, the cut-off Λ_i are needed. From (2.30), (2.31) and (2.32), we obtain the periods around the B-cycles,

$$\Pi_1 =\frac{1}{2}m_1\Lambda_1^3 - \frac{1}{2}(2S_1 - S_2)\left(1 - \log\frac{2S_1 - S_2}{\Lambda_1^3}\right) - \frac{1}{2}(2S_1 - S_2)\log m_1, \tag{2.35}$$

$$\begin{aligned}\Pi_j =&\frac{1}{2}m_j\Lambda_j^3 - \frac{1}{2}(2S_j - S_{j-1} - S_{j+1})\left(1 - \log\frac{2S_j - S_{j-1} - S_{j+1}}{\Lambda_j^3}\right)\\ &- \frac{1}{2}(2S_j - S_{j-1} - S_{j+1})\log m_j, \quad j = 2, \cdots, n-1,\end{aligned} \tag{2.36}$$

$$\begin{aligned}\Pi_n =&\frac{1}{2}m_n\Lambda_n^3 - \frac{1}{2}(2S_n - S_{n-1})\left(1 - \log\frac{2S_n - S_{n-1}}{\Lambda_n^3}\right)\\ &- \frac{1}{2}(2S_n - S_{n-1})\log m_n.\end{aligned} \tag{2.37}$$

We also calculate the periods around the A-cycles as

$$\mu_1 =\frac{1}{2}(2S_1 - S_2), \tag{2.38}$$

$$\mu_j =\frac{1}{2}(2S_j - S_{j-1} - S_{j+1}), \quad j = 2, \cdots, n-1, \tag{2.39}$$

$$\mu_n =\frac{1}{2}(2S_n - S_{n-1}). \tag{2.40}$$

Using these results and (1.2), we obtain the effective superpotential,

$$\begin{aligned}W_{\text{eff}} =&\sum_{i=1}^{n}\frac{1}{2}N_i m_i\Lambda_i^3\\ &- \frac{1}{2}\Bigg[N_1(2S_1 - S_2)\left(1 - \log\frac{2S_1 - S_2}{\Lambda_1^3}\right) - N_n(2S_n - S_{n-1})\left(1 - \log\frac{2S_n - S_{n-1}}{\Lambda_n^3}\right)\\ &- \sum_{i=2}^{n-1}N_i(2S_i - S_{i-1} - S_{i+1})\left(1 - \log\frac{2S_i - S_{i-1} - S_{i+1}}{\Lambda_i^3}\right)\Bigg]\\ &- \frac{1}{2}\Bigg[N_1(2S_1 - S_2)\log m_1 + N_n(2S_n - S_{n-1})\log m_n\\ &+ \sum_{i=2}^{n-1}N_i(2S_i - S_{i-1} - S_{i+1})\log m_i\Bigg]\\ &+ \pi i\Bigg[\tau_1(2S_1 - S_2) + \tau_n(2S_n - S_{n-1}) + \sum_{i=2}^{n-1}\tau_i(2S_i - S_{i-1} - S_{i+1})\Bigg].\end{aligned} \tag{2.41}$$

We can reproduce the Veneziano-Yankielowicz terms [50], which appear in the second term of (2.41). Note that, if we set all $N_i = N$, all $m_i = 1$, all $\Lambda_i = \Lambda$, all $S_i = S$ and all $\tau_i = \tau$, the effective superpotential is denoted simply by

$$W_{\text{eff}} = \frac{1}{2} n N \Lambda^3 - NS \left(1 - \log \frac{S}{\Lambda^3}\right) + 2\pi i \tau S. \tag{2.42}$$

Note that the constant terms $\sum_{i=1}^{n} \frac{1}{2} N_i m_i \Lambda_i^3$ in (2.41) and $\frac{1}{2} n N \Lambda^3$ in (2.42) can be ignored.

2.2. D_n singularity

Next let us consider the D_n singularities. A D_4 singularity appears in the compactifications of F-theory and is discussed also in the context of Dijkgraaf-Vafa conjecture [22].

The D_n quiver diagram is represented in fig. 3. In the same way as the A_n quiver gauge theories, we assign a $U(N_i)$ gauge group to each i-th node. The fields defined in the D_n quiver gauge theories are the adjoint scalars Φ_i for $i = 1, \cdots, n$ and the bifundamental matters Q_{ij} for the i-th and j-th nodes which are linked to each other.

The tree-level superpotential is

$$
\begin{aligned}
W(\Phi, Q) = &\sum_{i=1}^{n-1} \mathrm{Tr}(Q_{i,i+1}\Phi_{i+1}Q_{i+1,i} - Q_{i+1,i}\Phi_i Q_{i,i+1}) \\
&+ \mathrm{Tr}(Q_{n-2,n}\Phi_n Q_{n,n-2} - Q_{n,n-2}\Phi_{n-2}Q_{n-2,n}) + \sum_{i=1}^{n} \mathrm{Tr}W_i(\Phi_i).
\end{aligned} \tag{2.43}
$$

Integrating Q_{ij} out and rewriting the matrix integrals of Φ_i with the eigen value integrals of $\lambda_{i,I}$ which are the eigen values of Φ_i, we obtain the equations of motion for $\lambda_{i,I}$. From these equations of motion, we find the one-forms $y_i(x)dx$ of the deformed D_n singularity,

$$y_1(x) = W_1'(x) - 2S_1\omega_1(x) + S_2\omega_2(x), \tag{2.44}$$

$$y_j(x) = W_j'(x) - 2S_j\omega_j(x) + S_{j-1}\omega_{j-1}(x) + S_{j+1}\omega_{j+1}(x), \quad j = 2, \cdots, n-3, \tag{2.45}$$

$$
\begin{aligned}
y_{n-2}(x) = &W_{n-2}'(x) - 2S_{n-2}\omega_{n-2}(x) + S_{n-3}\omega_{n-3}(x) + S_{n-1}\omega_{n-1}(x) \\
&+ S_n\omega_n(x),
\end{aligned} \tag{2.46}
$$

$$y_{n-1}(x) = W_{n-1}'(x) - 2S_{n-1}\omega_{n-1}(x) + S_{n-2}\omega_{n-2}(x), \tag{2.47}$$

$$y_n(x) = W_n'(x) - 2S_n\omega_n(x) + S_{n-2}\omega_{n-2}(x), \tag{2.48}$$

by the use of the resolvents which are defined in the same way as (2.29). In the D_n case different from the A_n case, (2.46) is characteristic, because the $(n-2)$-th node is linked to the three nodes. The one-forms $y_i(x)dx$ include the quantum effects coming from the D_n matrix models.

Let us consider the quadratic superpotential (2.33). We can then calculate the periods around the B-cycles as

$$\Pi_1 = \frac{1}{2}m_1\Lambda_1^3 - \frac{1}{2}(2S_1 - S_2)\left(1 - \log\frac{2S_1 - S_2}{\Lambda_1^3}\right) - \frac{1}{2}(2S_1 - S_2)\log m_1, \tag{2.49}$$

$$
\begin{aligned}
\Pi_j = &\frac{1}{2}m_j\Lambda_j^3 - \frac{1}{2}(2S_j - S_{j-1} - S_{j+1})\left(1 - \log\frac{2S_j - S_{j-1} - S_{j+1}}{\Lambda_j^3}\right) \\
&- \frac{1}{2}(2S_j - S_{j-1} - S_{j+1})\log m_j, \quad j = 2, \cdots, n-3,
\end{aligned} \tag{2.50}
$$

$$\Pi_{n-2} =\frac{1}{2}m_{n-2}\Lambda_{n-2}^3 - \frac{1}{2}(2S_{n-2} - S_{n-3} - S_{n-1} - S_n)\left(1 - \log\frac{2S_{n-2} - S_{n-3} - S_{n-1} - S_n}{\Lambda_{n-2}^3}\right)$$
$$- \frac{1}{2}(2S_{n-2} - S_{n-3} - S_{n-1} - S_n)\log m_{n-2}, \tag{2.51}$$
$$\Pi_{n-1} =\frac{1}{2}m_{n-1}\Lambda_{n-1}^3 - \frac{1}{2}(2S_{n-1} - S_{n-2})\left(1 - \log\frac{2S_{n-1} - S_{n-2}}{\Lambda_{n-1}^3}\right)$$
$$- \frac{1}{2}(2S_{n-1} - S_{n-2})\log m_{n-1}, \tag{2.52}$$
$$\Pi_n =\frac{1}{2}m_n\Lambda_n^3 - \frac{1}{2}(2S_n - S_{n-2})\left(1 - \log\frac{2S_n - S_{n-2}}{\Lambda_n^3}\right) - \frac{1}{2}(2S_n - S_{n-2})\log m_n, \tag{2.53}$$

and the periods around the A-cycles as

$$\mu_1 =\frac{1}{2}(2S_1 - S_2), \tag{2.54}$$
$$\mu_j =\frac{1}{2}(2S_j - S_{j-1} - S_{j+1}), \quad j = 2, \cdots, n-3, \tag{2.55}$$
$$\mu_{n-2} =\frac{1}{2}(2S_{n-2} - S_{n-3} - S_{n-1} - S_n), \tag{2.56}$$
$$\mu_{n-1} =\frac{1}{2}(2S_{n-1} - S_{n-2}), \tag{2.57}$$
$$\mu_n =\frac{1}{2}(2S_n - S_{n-2}). \tag{2.58}$$

From these periods, we can obtain the effective superpotentials in terms of (1.2).

For example, we consider the D_4 quiver gauge theory. The effective superpotential becomes

$$\begin{aligned} W_{\text{eff}} = &-\frac{1}{2}(2S_1 - S_2)\left(1 - \log\frac{2S_1 - S_2}{\Lambda^3}\right) - \frac{1}{2}(2S_3 - S_2)\left(1 - \log\frac{2S_3 - S_2}{\Lambda^3}\right) \\ &- \frac{1}{2}(2S_4 - S_2)\left(1 - \log\frac{2S_4 - S_2}{\Lambda^3}\right) \\ &- \frac{1}{2}(2S_2 - S_1 - S_3 - S_4)\left(1 - \log\frac{2S_2 - 2S_1 - S_3 - S_4}{\Lambda^3}\right) \\ &+ \pi i\tau(S_1 + S_3 + S_4 - S_2), \end{aligned} \tag{2.59}$$

where, for simplicity, we set that all $m_i = 1$, all $N_i = N$, all $\Lambda_i = \Lambda$ and all $\tau_i = \tau$, and the constant term $\sum_i \frac{1}{2}N_i m_i\Lambda_i^3$ is ignored. Since the first, third and fourth nodes of the D_4 quiver diagram have a cyclic symmetry, S_1, S_3, S_4 in the superpotential (2.59) can be replaced with one another.

Finally we consider the E_n singularities. In the string theories, the E_n singularities play important roles. For example, $E_8 \times E_8$ gauge symmetry of heterotic string theories are realized by the E_n singular fibres in the F-theory. So it is interesting to analyse the E_n singularities.

The E_n quiver diagram is depicted in fig. 4. Each i-th node has a $U(N_i)$ gauge group. In the same way as A_n and D_n quiver gauge theories, we define the adjoint scalars Φ_i and the bifundamental matters Q_{ij}. The tree-level superpotential in the E_n quiver matrix models is

$$\begin{aligned}W(\Phi, Q) =& \sum_{i=1}^{n-1} \mathrm{Tr}(Q_{i,i+1}\Phi_{i+1}Q_{i+1,i} - Q_{i+1,i}\Phi_i Q_{i,i+1}) \\ &+ \mathrm{Tr}(Q_{3n}\Phi_n Q_{n3} - Q_{n3}\Phi_3 Q_{3n}) + \sum_{i=1}^{n} \mathrm{Tr}W_i(\Phi_i).\end{aligned} \tag{2.60}$$

We integrate Q_{ij} out in the partition function $Z = \int d\Phi dQ \exp\left(-\frac{1}{g_s}W(\Phi, Q)\right)$ and obtain the effective action of $\lambda_{i,I}$ which are the eigen values of Φ_i. We calculate the equations of motion from this effective action and read the following functions,

$$y_1(x) = W_1'(x) - 2S_1\omega_1(x) + S_2\omega_2(x), \tag{2.61}$$

$$y_2(x) = W_2'(x) - 2S_2\omega_2(x) + S_1\omega_1(x) + S_3\omega_3(x), \tag{2.62}$$

$$y_3(x) = W_3'(x) - 2S_3\omega_3(x) + S_2\omega_2(x) + S_4\omega_4(x) + S_n\omega_n(x), \tag{2.63}$$

$$\begin{aligned}y_j(x) =& W_j'(x) - 2S_j\omega_j(x) + S_{j-1}\omega_{j-1}(x) \\ &+ S_{j+1}\omega_{j+1}(x), \quad j = 4, \cdots, n-2,\end{aligned} \tag{2.64}$$

$$y_{n-1}(x) = W_{n-1}'(x) - 2S_{n-1}\omega_{n-1}(x) + S_{n-2}\omega_{n-2}(x), \tag{2.65}$$

$$y_n(x) = W_n'(x) - 2S_n\omega_n(x) + S_3\omega_3(x), \tag{2.66}$$

We also consider the quadratic superpotential (2.33) and assume the eigen values in $\omega_i(x)$ to be in the vacua, that is, $\lambda_{i,I} = 0$. Since every critical point of $W_i(x)$ is then splitted to the two points by the quantum deformations, we calculate the period (2.34) for the one-form $y_i(x)dx$. The periods Π_i around the B-cycles are

$$\Pi_1 = \frac{1}{2}m_1\Lambda_1^3 - \frac{1}{2}(2S_1 - S_2)\left(1 - \log\frac{2S_1 - S_2}{\Lambda_1^3}\right) - \frac{1}{2}(2S_1 - S_2)\log m_1, \tag{2.67}$$

$$\begin{aligned}\Pi_2 =& \frac{1}{2}m_2\Lambda_2^3 - \frac{1}{2}(2S_2 - S_1 - S_3)\left(1 - \log\frac{2S_2 - S_1 - S_3}{\Lambda_2^3}\right) \\ &- \frac{1}{2}(2S_2 - S_1 - S_3)\log m_2,\end{aligned} \tag{2.68}$$

$$\begin{aligned}\Pi_3 =& \frac{1}{2}m_3\Lambda_3^3 - \frac{1}{2}(2S_3 - S_2 - S_4 - S_n)\left(1 - \log\frac{2S_3 - S_2 - S_4 - S_n}{\Lambda_3^3}\right) \\ &- \frac{1}{2}(2S_3 - S_2 - S_4 - S_n)\log m_3,\end{aligned} \tag{2.69}$$

$$\begin{aligned}\Pi_j =&\frac{1}{2}m_j\Lambda_j^3 - \frac{1}{2}(2S_j - S_{j-1} - S_{j+1})\left(1 - \log\frac{2S_j - S_{j-1} - S_{j+1}}{\Lambda_j^3}\right)\\ &- \frac{1}{2}(2S_j - S_{j-1} - S_{j+1})\log m_j, \quad j = 4, \cdots, n-2, \end{aligned}\tag{2.70}$$

$$\begin{aligned}\Pi_{n-1} =&\frac{1}{2}m_{n-1}\Lambda_{n-1}^3 - \frac{1}{2}(2S_{n-1} - S_{n-2})\left(1 - \log\frac{2S_{n-1} - S_{n-2}}{\Lambda_{n-1}^3}\right)\\ &- \frac{1}{2}(2S_{n-1} - S_{n-2})\log m_{n-1}, \end{aligned}\tag{2.71}$$

$$\Pi_n =\frac{1}{2}m_n\Lambda_n^3 - \frac{1}{2}(2S_n - S_3)\left(1 - \log\frac{2S_n - S_3}{\Lambda_n^3}\right) - \frac{1}{2}(2S_n - S_3)\log m_n. \tag{2.72}$$

and the periods μ_i around the A-cycles are

$$\mu_1 =\frac{1}{2}(2S_1 - S_2), \tag{2.73}$$

$$\mu_2 =\frac{1}{2}(2S_2 - S_1 - S_3), \tag{2.74}$$

$$\mu_3 =\frac{1}{2}(2S_3 - S_2 - S_4 - S_n), \tag{2.75}$$

$$\mu_j =\frac{1}{2}(2S_j - S_{j-1} - S_{j+1}), \quad j = 4, \cdots, n-2, \tag{2.76}$$

$$\mu_{n-1} =\frac{1}{2}(2S_{n-1} - S_{n-2}), \tag{2.77}$$

$$\mu_n =\frac{1}{2}(2S_n - S_3). \tag{2.78}$$

From these periods and (1.2), we can calculate the effective superpotentials of the E_n quiver gauge theories. For simplicity, we ignore the terms independent of S_i and set that all $m_i = 1$, all $N_i = N$, all $\Lambda_i = \Lambda$ and all $\tau_i = \tau$. For example the effective superpotential of the E_8 quiver then becomes

$$\begin{aligned}W_{\text{eff}} =& -\frac{1}{2}N(2S_1 - S_2)\left(1 - \log\frac{2S_1 - S_2}{\Lambda^3}\right) - \frac{1}{2}N(2S_7 - S_6)\left(1 - \log\frac{2S_7 - S_6}{\Lambda^3}\right)\\ &- \frac{1}{2}N(2S_8 - S_3)\left(1 - \log\frac{2S_8 - S_3}{\Lambda^3}\right)\\ &- \frac{1}{2}N(2S_3 - S_2 - S_4 - S_8)\left(1 - \log\frac{2S_3 - S_2 - S_4 - S_8}{\Lambda^3}\right)\\ &- \frac{1}{2}N\sum_{j=2,4,5,6}(2S_j - S_{j-1} - S_{j+1})\left(1 - \log\frac{2S_j - S_{j-1} - S_{j+1}}{\Lambda^3}\right)\\ &+ \pi i\tau(S_1 + S_7 + S_8 - S_3).\end{aligned}$$

In this effective superpotential we can also find Veniziano-Yankielowicz terms. Since the third node is linked to the three nodes, S_3 is characteristic in the E_n quiver as well as in the D_n quiver.

3. Conclusions and discussions

We have considered the Calabi-Yau manifolds with the ADE singularities. If we calculate the periods of one-forms $y_i(x)dx$ around compact A-cycles and non-compact B-cycles on the deformed geometry of the Calabi-Yau manifolds, we can obtain the effective superpotentials of the dual gauge theories by the geometric engineering. We have calculated the equations of motion in the ADE matrix models. Since the quantum deformations are derived from the perturbative analyses of the matrix models by the Dijkgraaf-Vafa conjecture, We have found out the quantum deformations of the one-forms $y_i(x)dx$ from those equations of motions in the ADE matrix models.

We have considered the quadratic superpotentials $W_i(\Phi_i) = \frac{1}{2}m_i \mathrm{Tr}\Phi_i^2$ for the simple examples. Then the classical vacua are $\lambda_{i,I} = 0$, where $\lambda_{i,I}$ are the eigen values of Φ_i. Since the perturbation theory on these vacua gives rise to the effective superpotential in the dual gauge theory, we have approximated $\lambda_{i,I}$ in the resolvents ω_i to the vacua. The original critical point $a_i = 0$, which is obtained from $(y_i^{\mathrm{cl}} =)W_i'(a_i) = 0$, is splitted to the two points $a_i^{\pm}$, which are derived from $y_i(a_i^{\pm}) = 0$ on the deformed geometry. In terms of $a_i^{\pm}$ and the cut-off parameters Λ_i, we have calculated the periods (2.34). From these periods we have written down the effective superpotentials of the dual quiver gauge theories. We have also found that the effective superpotentials include the Veneziano-Yankielowicz terms.

We have used the approximation, that is, all eigen values of Φ_i appearing in the resolvents are in the classical vacua. But in order to derive exact effective superpotentials for the $\mathcal{N} = 1$ quiver gauge theories, we should achieve the integration of the eigen values in the multi-matrix model partition function.

Though it is difficult to exactly calculate the partition functions of the multi-matrix models, we can analyse the loop expansions of the planar diagrams order by order of the 't Hooft couplings S_i by the use of Feynman diagrams [18]. So we should confirm the expansions in the context of the geometric engineering.

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